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# Stokes Phenomena and Monodromy Deformation Problem for Nonlinear Schrödinger Equation

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Following Flaschka and Newell, the inverse problem for Painleve IV is formulated with the help of similarity variables. The Painleve IV arises as the eliminant of the two second-order ordinary differential equations originating from the nonlinear Schrödinger equation. Asymptotic expansions are obtained near the singularities at zero and infinity of the complex eigenvalue plane. The corresponding analysis then displays the Stokes phenomena. The monodromy matrices connecting the solution $Y_j$ in the sector $S_j$ to that in $S_{j+1}$ are fixed in structure by the imposition of certain conditions. It is then shown that a deformation keeping the monodromy data fixed leads to the nonlinear Schrödinger equation. While Flaschka and Newell did not make any absolute determination of the Stokes parameters, the present approach yields the values of the Stokes parameters in an explicit way, which in turn can determine the matrix connecting the solutions near zero and infinity. Finally, it is shown that the integral equation originating from the analyticity and asymptotic nature of the problem leads to the similarity solution previously determined by Boiti and Pampinelli.

## 1. INTRODUCTION

Recently two important but parallel theories have been developed for the complete analysis of nonlinear partial differential equations. One is the method of the inverse scattering transform (IST) (Eilenberger, 1981) and the other is that of monodromy deformation (MD) (Chudnovsky, 1982). While several authors have enriched the subject of IST, the contributions to the field of MD are relatively few. The only exhaustive approach is that of Kyoto school (Ueno and Date, 1973a,b; Jimbo and Miwa, 1980). Another approach is that of Flaschka and Newell (1979). While the method of the Japanese school is relatively abstract, being based on infinite-dimensional Lie algebra, that of Flaschka and Newell (FN) is more concrete and oriented

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to special nonlinear equations through its connection to the special class of Painleve equations. One elegant aspect of the approach of FN is that it exhibits very clearly how the asymptotic expansion can be used in conjunction with analyticity arguments to analyze the Stokes phenomena, and hence the monodromy deformation problem.

In this paper we make a departure from the treatment of FN. In their paper, the absolute determination of the Stokes parameters was not possible, but here we show that by taking account of a classical analysis of Sibuya, it is possible to find the explicit values of the Stokes constants. These values can then be used in the equations determining the matrix connecting the solution vector near zero and infinity, for their determination [see equation (31) in Section 3]. In this connection it can be noted that almost all the integrable nonlinear equations reduce to some Painleve transcendents through the similarity variables. On the other hand, the nonlinear Schrödinger equation reduces to a pair of coupled ordinary equations equivalent to the Painleve IV, as shown by Boiti and Pampinelli (1979, 1980a,b). Here we apply the methodology of FN, slightly amended by incorporating the theory of Sibuya, to the case of Painleve IV. At this point we may point out that though the work of Ueno and Date (1973a,b) and Jimbo and Miwa (1980) encompasses all the Painleve equations, the formalistic nature of their approach is quite difficult to appreciate in terms of the results of any particular equation. On the other hand, our approach is of a pedagogical nature and clearly indicates the ways to circumvent the difficulties encountered in an analysis of the problem.

## 2. FORMULATION

The nonlinear Schrödinger equation (NLSE) under consideration reads

$$iq_t - q_{xx} = \pm 2q^2 q^* \tag{1}$$

The AKNS inverse problem pertaining to equation (1) is

$$\begin{aligned} v_{1x} &= -i\zeta' v_1 + q v_2 \\ v_{2x} &= i\zeta' v_2 + r v_1 \end{aligned} \tag{2}$$

along with

$$\begin{aligned} v_{1t} &= A v_1 + B v_2 \\ v_{2t} &= C v_1 + D v_2 \end{aligned} \tag{3}$$

where $A$, $B$, $C$, and $D$ are well-known functions of $q$, $r$, $\varphi'$, and for the NLSE we assume $q = r^*$. The similarity variable, which can be found either by a Lie point symmetry analysis or by a scaling argument, is given as

$$z = xt^{-1/2}; \qquad q(x, t) = t^{1/2}\phi(x/t^{1/2})$$

We then convert equation (1) to the ordinary nonlinear coupled system of differential equations

$$-\frac{d}{dz}\left(\phi_z+\frac{iz}{2}\phi\right)=\pm 2\phi^2\phi^* \tag{4}$$

The main trick of FN is to convert the lax pair (2) and (3) to such variables, for which we set

$$v=\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}; \qquad v=v(xt^{-1/2},\zeta' t^{1/2})=v(z,\zeta)$$

so that we have

$$\begin{aligned} v_z^1 &= -i\varphi v^1+\phi v^2 \\ v_z^2 &= i\varphi v^2+\phi^* v^1 \end{aligned} \tag{5}$$

$$\begin{aligned} v_\varphi^1 &= \left(4i\varphi+\frac{2i\phi\phi^*}{\varphi}-iz\right)v^1+\left(-4\phi-\frac{2i\phi_z}{\varphi}+\frac{z\phi}{\varphi}\right)v^2 \\ v_\varphi^2 &= \left(-4\phi^*+\frac{2i\phi_z^*}{\varphi}+\frac{z\phi^*}{\varphi}\right)v^1+\left(-4i\varphi-\frac{2i\phi\phi^*}{\varphi}+iz\right)v^2 \end{aligned} \tag{6}$$

so that in matrix form we can set

$$v_\varphi=\left[A_0\varphi+A_1+\frac{1}{\varphi}A_2\right]v \tag{7}$$

where

$$A_0=\begin{pmatrix} 4i & 0 \\ 0 & -4i \end{pmatrix}, \qquad A_1=\begin{pmatrix} -iz & -4\phi \\ -4\phi^* & iz \end{pmatrix}$$

$$A_2=2i\begin{pmatrix} \phi\phi^* & -(\phi_z+\frac{1}{2}iz\phi) \\ (\phi_z^*-\frac{1}{2}iz\phi^*) & -\phi\phi^* \end{pmatrix}$$

Equations (5) and (7) form the basis of the asymptotic expansion that we perform in the next section.

## 3. ASYMPTOTIC EXPANSION

For the construction of the asymptotic expansion, we set near $\zeta=0$

$$v=[\exp(a_0\varphi^2+a_1\varphi)]\varphi^\mu \sum c_k\varphi^{-k} \tag{8}$$

so that

$$v_\varphi=\left(2a_0\varphi+a_1+\frac{\mu-k}{\varphi}\right)\varphi^\mu \exp(a_0\varphi^2+a_1\varphi)\sum c_k\varphi^{-k}$$

Then from equation (7) we get

$$\left(2a_0\varphi + a_1 + \frac{\mu - k}{\varphi}\right)[\exp(a_0\varphi^2 + a_1\varphi)]\varphi^\mu \sum c_k\varphi^{-k}$$
$$= (A_0\varphi + A_1 + A_2\varphi^{-1})[\exp(a_0\varphi^2 + a_1\varphi)]\varphi^\mu \sum c_k\varphi^{-k} \tag{9}$$

Equating different powers of $\varphi$ in (9), we construct equations for $c_k$, which can be solved to yield the two independent sets of solutions

$$\tilde{v}_\infty(1, z, \varphi) \sim \exp(2i\varphi^2 - iz\varphi)$$
$$\times \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \varphi^{-1} \begin{pmatrix} \phi\phi_z^* - \phi^*\phi_z - \frac{1}{2}iz\phi\phi^* \\ \frac{1}{2}i\phi^* \end{pmatrix} + \cdots \right\} \tag{10}$$

$$\tilde{v}_\infty(2, z, \varphi) \sim \exp(-2i\varphi^2 + iz\varphi)$$
$$\times \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \varphi^{-1} \begin{pmatrix} -\frac{1}{2}i\phi \\ \phi^*\phi_z - \phi\phi_z^* + \frac{1}{2}iz\phi\phi^* \end{pmatrix} + \cdots \right\} \tag{11}$$

At this point it is interesting to note an identity that will be useful later. From equation (4) we note that

$$\phi\phi_z^* - \phi^*\phi_z - \frac{iz}{2}\phi\phi^* = \frac{i}{2}\int \phi\phi^*\, dz + c \tag{12}$$

To obtain the asymptotic expansion near $\varphi = 0$, we put $\varphi = 1/\eta$ and let $\eta \to \infty$ in equation (7). Then (7) is transformed to

$$V_\eta = -[A_2\eta^{-1} + A_1\eta^{-2} + A_0\eta^{-3}]V \tag{13}$$

We then set

$$V = \eta^\mu \sum c_k\eta^{-k}$$

so that we get

$$\frac{\mu - k}{\eta}\eta^\mu \sum c_k\eta^{-k} = -(A_2\eta^{-1} + A_1\eta^{-2} + A_0\eta^{-3})\eta^\mu \sum c_k\eta^{-k}$$

Then the degeneracy condition for $c_0$ leads to the following equation for $\mu$:

$$\det[A_2 + I\mu] = 0 \tag{14}$$

from which we obtain

$$\mu^2 = 4\left[\left(\phi_z + \frac{iz}{2}\phi\right)\left(\phi_z^* - \frac{iz}{2}\phi^*\right)\right] - (\phi\phi^*)^2 \tag{15}$$

But we observe an important fact:

$$\frac{d\mu^2}{dz} = 0 \qquad \text{if} \quad \phi\phi^* = \text{const} \tag{16}$$

So the set of solutions near $\varphi=0$ is

$$\tilde{v}_0(1, z, \varphi)$$
$$= e^{u(x)}\varphi^{-2k}\left\{(i(\phi_z+\tfrac{1}{2}iz\phi)/(k+i\phi\phi^*))+\frac{\varphi^{-1}}{1-4k}\begin{pmatrix}x_2\\ y_2\end{pmatrix}+\cdots\right\}$$

where

$$\begin{pmatrix}x_2\\ y_2\end{pmatrix}=\begin{pmatrix}i(\phi_z+\frac{1}{2}iz\phi)/(i\phi\phi^*-k)\\ 1\end{pmatrix}z$$

$$z=-2i\left(\phi_z^*-\frac{iz}{2}\phi^*\right)\left(4\phi-\frac{z(\phi_z+\frac{1}{2}iz\phi)}{i\phi\phi^*+k}\right)-(2i\phi\phi^*+2k-1)$$
$$\times\left(iz-4i\phi^*\frac{\phi_z+\frac{1}{2}iz\phi}{i\phi\phi^*+k}\right) \tag{17}$$

and

$$\tilde{v}_0(2, z, \varphi)\approx e^{u^*(z)}\varphi^{2k}\left\{\begin{pmatrix}1\\ i(\phi_z+\frac{1}{2}iz\phi)/(\frac{1}{2}\phi\phi^*-k)\end{pmatrix}+\cdots\right\} \tag{18}$$

where $u(z)$ is the normalizing factor for the solution near $\varphi=0$, and is given as

$$u=\int\phi\,dz;\qquad u^*=\int\phi^*\,dz$$

The factors in the above expressions can be simplified if we use equation (15), but we prefer to keep the general structure. At this point we mention some important features of equations (5) and (6):

1. If $v(1, \varphi, z)$ is a solution, then $Mv^*(1, \varphi^*, z)$ is also a solution.
2. If $M\tilde{v}^*(2, \varphi^*, z)$ is a solution, then $\tilde{v}(2, \varphi, z)$ is another solution, where $M=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, and $v(n, \varphi, z)$ denotes the solution vector $(v_1, v_2)$, with $n$ indicating the first or second type of solution.

## 4. REGIONS OF GROWTH AND DECAY

The next step in our analysis is the segregation of zones in the complex $\varphi$ plane, where the solutions defined in Section 2 show a definite pattern of dominancy or subdominancy. From expressions (10) and (11) we can make the important inferences listed in Table I. Figure 1 shows this division of the complex eigenvalue plane into several sectors. The lines in the $\varphi$

Table I

| | | |
|---|---|---|
| $0 \le \arg \varphi < \pi/2$ | $v^1$ large | $v^2$ small |
| $\pi/2 \le \arg \varphi < \pi$ | $v^1$ small | $v^2$ large |
| $\pi \le \arg \varphi < 3\pi/2$ | $v^1$ large | $v^2$ small |
| $3\pi/2 \le \arg \varphi < 2\pi$ | $v^1$ small | $v^2$ large |
| $2\pi \le \arg \varphi < 2\pi + \delta$ | $v^1$ large | $v^2$ small |

plane originating from the origin on which the solution is maximally dominant or recessive are called Stokes lines. In the above situation

$$\arg \varphi = \pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4$$

are Stokes lines and the sectors are defined as

$$S_j = \left\{ \varphi \left| \begin{matrix} |\varphi| > \rho \\ j - 1/2\pi \le \arg \varphi < j\pi/2 \end{matrix} \right. \quad \text{for some } \rho \text{ with } j = 1, 2 \right\}$$

The anti-Stokes lines are

$$\arg \varphi = 0,\ \pi/2,\ \pi, 3\pi/2, 2\pi$$

In the above and also in what follows we use the following notation: $v_j^{(i)}(k, \varphi, z)$ denotes a solution of the linear equations (2) and (3), where

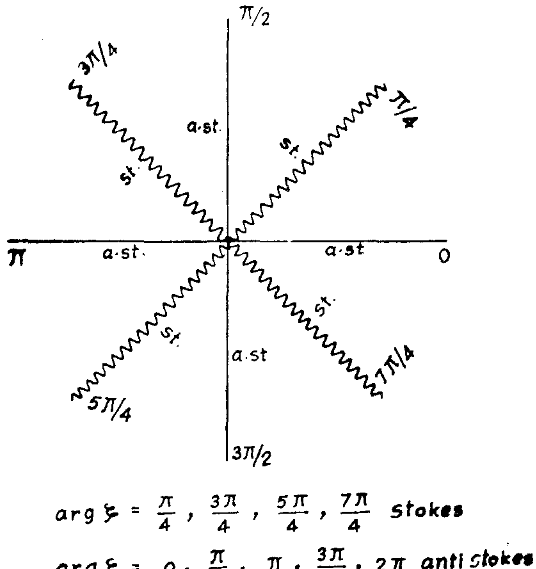


Fig. 1.

$(i)$ denotes the first or second component as above, $(j)$ denotes the sector, and $(k)$ denotes the type of solution. In general we have two types of solutions for our $2\times 2$ matrix system.

The next important step is to write down the basic form of the matrix or matrices connecting the solution vectors in several sectors. For this it is important to observe that a solution that was dominating in one sector may become subdominant when its leading terms are canceled by the contribution from the other component in the other sector. This fact dictates that the connection matrices are all triangular. Explicitly, we have

$$\begin{aligned} v_2 &= v_1\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \\ v_3 &= v_2\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \\ v_4 &= v_3\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \\ v_5 &= v_4\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, \qquad \text{but } v_5 \equiv v_1 \end{aligned} \tag{19}$$

Now utilizing the symmetry properties noted after equation (18) and taking account of (19), we get

$$\begin{aligned} v_2^{(1)} &= v_1^{(1)} + a v_1^{(2)} \\ v_4^{(1)} &= v_3^{(1)} + c v_3^{(2)} \end{aligned} \tag{20}$$

and so

$$c = a; \qquad d = b$$

for $\pi \leq \arg\, \varphi < 3\pi/2$,

$$v_2^{(1)} = b v_1^{(1)} + (1+ab) v_1^{(2)} \tag{21}$$

for $3\pi/2 \leq \arg\, \varphi < 2\pi$,

$$v_4(\varphi, z) = v_3(\varphi, z)\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \tag{22}$$

from which it follows that

$$\begin{aligned} v_4^{(1)} &= v_1^{(1)}(1+bc) + (a+c+abc) v_1^{(2)} \\ v_4^{(2)} &= b v_1^{(1)} + (1+ab) v_1^{(2)} \end{aligned}$$

Now crossing this zone, we come back to the first sector again, hence

$$v_5^{(1)} = v_4^{(1)} = (1+bc)v_1^{(1)} + (a+c+abc)v_1^{(2)}$$
$$v_5^{(2)} = dv_4^{(1)} + v_4^{(2)}$$
$$= v_1^{(1)}[d(1+bc)+d] + [1+ab+d(a+c+abc)]v_1^{(2)} \tag{23}$$

These relations will be of much use when we connect the solution near the origin to that at infinity.

Now from equation (17) we observe that

$$\tilde{v}(1, \varphi, z) = v(1, \varphi, z) - jiv(2, \varphi, z) \ln \varphi \tag{24}$$

where

$$j = \left[2\left(\phi_z^* - \frac{iz}{2}\phi^*\right)\left(\phi - z\frac{\phi_z + \frac{1}{2}iz\phi}{i\phi\phi^* - k}\right) - (2i\phi\phi^* + 2k - 1) \times \left(iz - 4i\phi^* \frac{\phi_z + \frac{1}{2}iz\phi}{i\phi\phi^* + k}\right)\right] e^{u-u^*}.$$

The logarithm will disappear if $j = 0$ and $k = \frac{1}{4}$. With the help of the sector relations we obtain from (24)

$$\begin{aligned} M\tilde{v}(1, \varphi\, e^{-i\pi}, z) &= e^{2i\pi k} v(1, \varphi, z) - \pi j\, e^{-2\pi i k} v(2, \varphi, z) \\ M\tilde{v}(2, \varphi\, e^{-i\pi}, z) &= e^{-2\pi i k} v(2, \varphi, z) \end{aligned} \tag{25}$$

so if in the sector $0 \le \arg \varphi < 2\pi$ the solution is $\tilde{v}$, then $\tilde{v}(\varphi\, e^{2\pi i}, z) = v(\varphi, z)J$ is a fundamental solution in $(2\pi, 4\pi)$, where

$$J = \begin{pmatrix} e^{-4i\pi k} & 0 \\ 2\pi j\, e^{4i\pi k} & e^{4\pi i k} \end{pmatrix} \tag{26}$$

It is interesting to note that det $J = 1$ for all $k$. We now seek the matrix connecting $v_0$ to $v_\infty$ as

$$v_\infty = v_0 A$$

with (27)

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

Now

$$\det v_\infty = 1, \qquad \det v_0 = \frac{-2ik(\phi_z + \frac{1}{2}iz\phi)}{(i\phi\phi^* + k)(i\phi\phi^* - k)}$$

so that

$$\det A = \frac{(i\phi\phi^* + k)(i\phi\phi^* - k)}{-2ik(\phi_z + \frac{1}{2}iz\phi)}$$

Then ($a$, $b$, $c$, $d$, $\alpha$, $\beta$, $\gamma$, $\delta$, det $v_\infty = 1$, and the coefficients of the asymptotic expansions) form the monodromy data for our system.

## Properties of the Matrix $A$

Now it follows from equation (27) that

$$v_\infty(\varphi\, e^{2\pi i}) = v_0(\varphi\, e^{2\pi i})A = v_0(\varphi)JA \tag{28}$$

and in the last sector

$$v_5(\varphi\, e^{2\pi i}) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} v_1(\varphi\, e^{2\pi i})$$

leading to

$$\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = A^{-1}J^{-1}A \tag{29}$$

We now set $\varphi = \hat{\varphi}\, e^{-i\pi}$ in the solution $v_\infty$ and apply $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, taking account of the relations (20)-(23), and obtain the important and fundamental relations

$$\begin{aligned} Mv^{(1)} &= (\alpha\delta\, e^{2\pi ik} + \alpha\beta\pi j\, e^{-2\pi ik} - \beta\gamma\, e^{-2\pi ik})v^{(1)} \\ &\quad + \left(-2\alpha\gamma \frac{e^{2\pi ik} - e^{-2i\pi k}}{2} - \alpha^2\pi j\, e^{-2\pi ik}\right)v^{(2)} \\ Mv^{(2)} &= \left(2\rho\delta \frac{e^{2\pi ik} - e^{-2\pi ik}}{2} + \beta^2\pi j\, e^{-2\pi ik}\right)v^{(1)} \\ &\quad + (-\beta\gamma\, e^{2\pi ik} - \alpha\beta\pi j\, e^{-2\pi ik} + \delta\alpha\, e^{-2\pi ik})v^{(2)} \end{aligned} \tag{30}$$

from which we deduce

$$\begin{aligned} 1 &= \alpha\delta\, e^{2\pi ik} + \alpha\beta\pi j\, e^{-2\pi ik} - \beta\gamma\, e^{-2\pi ik} \\ a &= -2\alpha\gamma \sin(2\pi k) - \pi j\alpha^2\, e^{-2\pi ik} \\ b &= \sin(2\pi k) + \pi j\beta^2\, e^{-2\pi ik} \\ 1 + ab &= -\beta\gamma\, e^{2\pi ik} - \alpha\beta\pi j\, e^{-2\pi ik} + \alpha\delta\, e^{-2\pi ik} \end{aligned} \tag{31}$$

Until now we have been following the methodology as laid down in Flaschka and Newell (1979). But as can be seen clearly, although this approach helps to get the properties of the monodromy data, it is not possible to determine the Stokes matrices absolutely. In the next section we show that by taking account of a paper by Sibuya we can explicitly determine the Stokes matrices, which in turn can lead to a complete determination of ($\alpha$, $\beta$, $\gamma$, $\delta$), the matrix connecting the solutions near zero to that at infinity.

## 5. SIBUYA'S APPROACH TO THE STOKES PARAMETERS

Sibuya (1977; also see Wasow, 1976) treats the equation

$$\frac{d^2v}{d\varphi^2}-(\varphi^{\mu}+a_1\varphi^{\mu-1}+\cdots+a_{\mu-1}\varphi+a_{\mu})\nu=0 \tag{32}$$

under the following assumptions:

1. The differential equation (32) has the unique solution

$$v=v_{\mu}(\varphi, a_1, \ldots, a_{\mu})$$

2. $v$ is an entire function of the parameters $(\varphi, a_1, a_2, \ldots, a_{\mu})$
3. $v$ admits an asymptotic representation

$$v\sim\varphi^{\gamma_{\mu}}\left(1+\sum_{n=1}^{\infty} B_{\mu,n}\varphi^{-n/2}\right)\exp[-iE_{\mu}(\varphi, t)]$$

as $\varphi$ tends to infinity in the different sectors, where $E_{\mu}(\varphi, t)$ is represented as

$$E_{\mu}(\varphi, t)\approx\frac{2}{\mu+2}\varphi^{(\mu+2)/2}+\sum_{n=1}^{\mu+1} A_{\mu,n}\varphi^{(\mu+2-n)/2} \tag{33}$$

and $\gamma_{\mu}$, $A_{\mu n}$, and $B_{\mu,n}$ are polynomials in $(a_1, \ldots, a_{\mu})$. Now if we put

$$(1+a_1\varphi^{-1}+\cdots+a_{\mu}\varphi^{-\mu})^{1/2}=1+\sum_{k=1}^{\infty} b_k\varphi^{-k}$$

then the quantities $\gamma_{\mu}$ and $A_{\mu,n}$ are given by

$$\gamma_{\mu}=\begin{cases}-\mu/4 & \mu \text{ odd}\\ -\mu/4-b\mu/2+1 & \mu \text{ even}\end{cases}$$

and

$$\sum_{n=1}^{\mu+1} A_{\mu,n}\varphi^{(\mu+2-n)/2}=\sum\frac{2}{\mu+2-2n}b_n\varphi^{(\mu+2-2n)/2}, \qquad 1\leq n\leq\frac{\mu}{2}+1 \tag{34}$$

4. If we choose $\phi$ such that $\exp[i(\mu+2)\phi]=1$, then the function $v(\hat{\varphi}, e^{i\phi}a_1\cdots e^{i\phi\mu}a_{\mu})$ is also a solution.

With

$$\theta=\exp[i2\pi/(\mu+2)]$$

the solution in the $J$th sector is given as

$$v_j(\varphi, t)\sim\theta^{-j\gamma_{\mu-j}}\varphi^{\gamma_{\mu j}}\left(1+\sum_{n=1}^{\infty} B_{\mu,n,j}\ \varphi^{-n/2}\right)$$
$$\times\exp[(-1)^{j+1}iE_{\mu}(\varphi, \alpha)] \tag{35}$$

as $\varphi\to\infty$ in the sectors.

5. The two solutions $v_{\mu,j+1}$ and $v_{\mu,j+2}$ are linearly independent because $v_{\mu,j+1}$ is subdominant in the $(j+1)$th sector and $v_{\mu,j+2}$ is dominant. Therefore, $v_\mu$ is a linear combination of $v_{j+1}^{(\mu)}$ and $v_{j+2}^{(\mu)}$,

$$v_j(\varphi, t) = c_j(t) v_{j+1} + \tilde{c}_j(t) v_{j+2} \tag{36}$$

$c_j$ and $\tilde{c}_j$ are Stokes multipliers. For $\mu = 2$

$$c_j(a_1, a_2) = \begin{cases} 2^{b_2} \exp\left[\frac{1}{4} a_1^2 - i\pi\left(\frac{b_2}{2} - \frac{1}{4}\right)\right] \frac{\sqrt{2}\pi}{\Gamma(\frac{1}{2} + b_2)}, & j \text{ even} \\ 2^{-b_2} \exp\left[-\frac{1}{4} a_1^2 + i\pi\left(\frac{b_2}{2} + \frac{1}{4}\right)\right] \frac{\sqrt{2}\pi}{\Gamma(\frac{1}{2} - b_2)}, & j \text{ odd} \end{cases}$$
$$\tilde{c}_j = \begin{cases} -i\, e^{-\pi b_2} & j \text{ even,} \\ -i\, e^{\pi b_2} & j \text{ odd,} \end{cases} \qquad b_2 = \frac{1}{2a_2} - \frac{1}{8} a_1^2 \tag{37}$$

We have quoted the above result for the sake of clarity. To apply the above result we first single second-order equation by an ordinary single second-order equation by eliminating any one of the components.

By eliminating $v_2$, we get

$$v_{1\varphi\varphi} = \left\{ -16\varphi^2 + 8z\varphi - 4i - z^2 + \frac{1}{\varphi}\left[ 8i(\phi^*\phi_z - \phi\phi_z^*) - 4z\phi\phi^* \right] \right.$$
$$\left. + \frac{1}{\varphi^2}\left[ -2i\phi\phi^* - \phi^2\phi^{*2} + \left(\phi_z + \frac{iz}{2}\phi\right)\left(\phi_z^* - \frac{iz}{2}\phi^*\right)\right] + \cdots \right\} \tag{38}$$

where we have not written down the terms represented by the dots, because they will not be important for $\varphi \to \infty$.

Scaling the variables $\varphi$ and $v$ as $2\varphi = \varphi'$ and $4v = v'$, we arrive at

$$v'_{1\varphi'\varphi'} = \left[ \varphi'^2 + z\varphi' - \left(i - \frac{z^2}{4}\right) \right.$$
$$\left. - 4\varphi'^{-1}\left(\phi\phi_z^* - \phi^*\phi_z - \frac{iz}{2}\phi\phi^*\right) + \cdots \right] v'_1 \tag{39}$$

Since this is a scalar equation, we will omit the index 4′ and write $v$. The index $j$ in $v_{(j)}$ will denote the sector of the solution as described in Table I. We will now utilize the results of Sibuya for equations of the type (39), and for that the identification of solutions of (39) with those of (6) is essential. If this is done, then equation (36) along with (37) will yield the Stokes parameters. To proceed with the program outlined above, we first switch from the vector to the matrix notation for the solutions in (10) and (11).

The matrix solution is constructed as

$$v^{ij} = \begin{pmatrix} v^{11} & v^{12} \\ v^{21} & v^{22} \end{pmatrix} \tag{40}$$

where

$$\begin{pmatrix} v^{11} \\ v^{21} \end{pmatrix}$$

is actually $\tilde{v}(1, z, \varphi)$ and

$$\begin{pmatrix} v^{12} \\ v^{22} \end{pmatrix}$$

is equivalent to $\tilde{v}(2, z, \varphi)$, in both the exact and the asymptotic situations. As before, the index $(j)$ in

$$\begin{pmatrix} v^{11}_{(j)} \\ v^{21}_{(j)} \end{pmatrix}$$

will denote the solution in the $j$th sector.

Now in our particular case we have

$$v^{11}_{(1)} = v^{11}_{(0)} + a v^{12}_{(0)} \tag{41}$$

and from equation (36)

$$v_{(-1)} = c_{-1} v_{(0)} + \tilde{c}_{-1} v_{(1)} \tag{42}$$

or

$$v_{(1)} = \frac{1}{\tilde{c}_{-1}} v_{(-1)} - \frac{c_{-1}}{\tilde{c}_{-1}} v_{(0)}$$

But we have the identification $v^{11}_{(0)} = v_{(-1)}$, $v^{11}_{(1)} = v_{(1)}$, and $v^{12}_{(0)} = (-i\phi) v_{(0)}$. These equations, when coupled with (42) and (41), yield

$$a = \frac{1}{i\phi} c_{-1}; \qquad \tilde{c}_{-1} = 1 \tag{43}$$

Furthermore,

$$v^{12}_{(2)} = v^{12}_{(1)} + b v^{11}_{(1)} \tag{44}$$

But

$$v_{(2)} = \frac{1}{i\phi} v^{12}_{(1)}, \qquad v_{(3)} = v^{11}_{(1)}, \qquad v_{(4)} = v^{12}_{(2)} \tag{44a}$$

Now

$$v_{(2)} = c_2 v_{(3)} + \tilde{c}_2 v_{(4)} \tag{45}$$

Comparing (44) and (45) with the help of (44a), we get

$$b = -i\phi c_2, \qquad \tilde{c}_2 = -1 \tag{46}$$

Similarly we deduce

$$\begin{aligned} c &= \frac{1}{2\phi} c_{-1}, \qquad \tilde{c}_{-1} = 1 \\ d &= -i\phi c_6, \qquad \tilde{c}_6 = -1 \end{aligned} \tag{47}$$

where the $c_j$ are given by equation (37) with

$$b_1 = -\frac{z}{2}, \qquad b_2 = \frac{1}{2}\left(i - \frac{z^2}{2}\right)$$

It is quite important to observe that our explicit determination of the Stokes parameters respects our earlier derived constraints, $a = c$ and $b = d$. Furthermore, if these explicit values are used in (31), then it is in principle possible to determine $(\alpha, \beta, \gamma, \delta)$, which was not the case of the Flaschka-Newell approach.

## 6. PROPERTIES OF THE MONODROMY DATA

1. The matrix functions $v_j$ are holomorphic in

$$S_j = \left\{ \varphi,\ |\varphi| > 0 \text{ and } \frac{j-1}{2}\pi \le \arg\varphi < \frac{\pi j}{2} \right\}$$

such that

$$\begin{aligned} v_j \sim \tilde{v}_j &= \left(1 + \frac{c_1}{\varphi} + \cdots\right)\begin{pmatrix} e^{\theta} & 0 \\ 0 & e^{-\theta} \end{pmatrix} \\ \theta &= 2i\varphi^2 - iz\varphi \qquad \text{as} \quad |\varphi| \to \infty \quad \text{in } S_j \end{aligned} \tag{48}$$

and $v_{j+1} = v_j A_j$.

2. A matrix function $\omega$ of the form

$$\omega(\varphi) = \hat{\omega}(\varphi)\begin{pmatrix} \varphi^{-2k} & 0 \\ 0 & \varphi^{2k} \end{pmatrix} \tag{49}$$

with $\hat{w}(\varphi)$ holomorphic, such that for $\varphi \in S_1$, $v_1(\varphi) = \omega(\varphi)A$, with

$$\det A = \frac{(i\phi\phi^* + k)(i\phi\phi^* - k)}{-2ik(\phi_z + \frac{1}{2}iz\phi)} \tag{50}$$

3. The solution matrix $v(\varphi)$ has the symmetry

$$M\tilde{v}^*(\varphi)M = \tilde{v}(\varphi), \qquad M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{51}$$

4. $A_j$ matrices are independent of $Z$. So, differentiating $v_{j+1} = v_j A_j$ with respect to $z$ and multiplying by $v_{j+1}^{-1}$, we get

$$\begin{aligned} v_{j+1z} v_{j+1}^{-1} &= v_{jz} A_j v_{j+1}^{-1} \\ &= v_{jz} A_j (v_j A_j)^{-1} = v_{jz} v_j^{-1} \end{aligned} \tag{52}$$

so that $v_{jz} v_j^{-1}$ is well defined in a deleted neighborhood of $\varphi = \infty$ and its asymptotic expansion is that of $\tilde{v}_z v^{-1}$ uniformly for $|\varphi| > \rho$. Now using (48) and its $z$ derivative, we have

$$\begin{aligned} \tilde{v}_z v^{-1} &= i\varphi \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + i\left[c_1, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right] \\ &= \begin{pmatrix} -i\varphi & \phi \\ \phi^* & i\varphi \end{pmatrix} \end{aligned} \tag{53}$$

which is nothing but the matrix occurring in the $L$ operator pertaining to the NLSE. Similarly, near $\varphi = 0$, we have

$$\omega_z \omega^{-1} = \tilde{\omega}_z \tilde{\omega}^{-1} = \phi(\varphi)$$

with

$$\tilde{v} = \begin{pmatrix} 1 - \dfrac{i}{2\varphi}\rho & -\dfrac{i\phi}{2\varphi} \\ \dfrac{i\phi^*}{2\varphi} & 1 + \dfrac{i\rho}{2\varphi} \end{pmatrix} \begin{pmatrix} e^{\theta} & 0 \\ 0 & e^{\theta *} \end{pmatrix} \tag{54}$$

Evaluating $\tilde{v}_\varphi \tilde{v}^{-1}$, we observe that $\tilde{v}_\varphi \tilde{v}^{-1}$ is equal to the time part of the Lax operator. Thus it is enough to demonstrate that the nonlinear equation is a result of isomonodromy deformation of the linear problem.

Writing out the contour integrals over the contours shown in Figure 2, we can prove the following [we do not give the details of the computation, since the considerations of Ablowitz (1983) remain almost unaltered):

$$\begin{aligned} \phi^* &= -\lim_{\varphi \to \infty} 2i\varphi v^2 e^{-\theta} \\ \phi &= \lim_{\varphi \to \infty} 2i\varphi v^2 e^{\theta} \end{aligned} \tag{55}$$

and finally we obtain

$$\phi = -\frac{1}{\pi} \int_{c_{13}} \exp(4i\varphi^2 - 2i\varphi z)\, d\varphi = \exp(-iz^2/2) \tag{56}$$

which satisfies both equations (6) and (7).

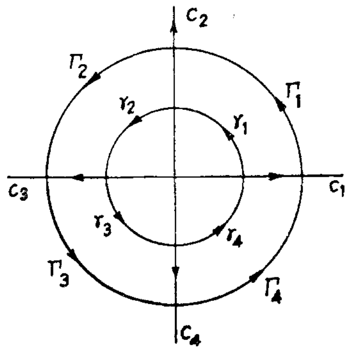


Fig. 2. Contour for the integral equation.

## 7. CONCLUSIONS

We have studied in detail the monodromy problem related to the nonlinear Schrödinger equation and Painleve IV, through similarity variables. Though the general problem of the deformation of second- and third-order equations has been studied by the Japanese school, we think that the above analysis helps to clarify special features that may arise in particular nonlinear problems.

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